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New Approach for Estimating the Two Parameter Gamma Distribution

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ABSTRACT: The two-parameter gamma distribution is widely used in statistical applications. Several methods to estimate its parameters have been proposed in the literature. Some of these methods do not have closed forms while others do. In this paper a closed formula is presented and compared with method of moments, maximum likelihood method and some other methods through simulation study. The results showed that the proposed method efficiently estimated the parameters and performed better.

الملخص: يعد توزيع غاما ثنائي المعلمة من التوزيعات الشائعة الاستخدام في التطبيقات الإحصائية. وقد تم اقتراح عدة طرق لتقدير معلماته في الأدبيات العلمية. بعض هذه الطرق لا تمتلك صيغًا مغلقة، بينما يمتلك بعضها الآخر ذلك. في هذه الورقة، يتم تقديم صيغة مغلقة ومقارنتها بطريقة العزوم، وطريقة الإمكان الأعظم، وبعض الطرق الأخرى من خلال دراسة محاكاة. وقد أظهرت النتائج أن الطريقة المقترحة تقدّر المعلمات بكفاءة وتظهر أداء أفضل.

I. INTRODUCTION

The Gamma distribution is an important distribution which is widely used in many fields such as environmental science, finance, survival analysis, lifetime data and reliability. Hence, it is very important to estimate the parameters efficiently. Several methods of estimating the two parameters have been proposed in the literature. The well-known methods are Maximum likelihood (ML) and method of moments (MM). The former method, unfortunately had no closed form expression while the latter despite of having a closed-forms they are not efficient under either small samples or large samples. Various studies have proposed different methods to estimate the parameters. Dang and Weerakkody (2000) derive bounds for the MLE of the shape parameter to develop a new algorithm to estimate it. Hwang and Huang (2001) proposed more convenient moment estimators by using the characterization of gamma distribution and shown to be more efficient than MLE and MM for small samples. While Pradhan and Kundu (2011) considered Bayes estimates based on the assumption that the scale parameter has a gamma prior and the shape parameter has any log-concave prior, and they are independently distributed. Ye & Chen (2016) obtained a closed form to estimate the parameters based on the idea that gamma distribution as a special case of generalized gamma distribution, which is denoted as MLEcf in the paper. On the other hand, Khamkong (2018) proposed an alternative closed form estimator based on transforming gamma distribution to a normal distribution (MLEc). Ke, Wang, Zhou and Ye (2023) proposed new approaches to parameter estimation using mean squared error representative points (MSE-RPs). Junmei and Liqin

(2024) presents an easy computation of the MLEs by modifying the MLE of the shape parameter to be more efficient than the existing estimators.

In this paper we propose a new approach to estimate the parameters of gamma distribution. Also a comparison between the new approach and some previous methods will be presented through simulation study.

The new approach

The gamma distribution denoted as $Ga(\alpha,\beta)$, is a two-parameter distribution with probability density function (pdf).

$$f(x)=\frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}x^{\alpha-1}\,e^{-\frac{1}{\beta}x}, \qquad x>0,\ \alpha,\beta>0$$

Where α is the shape parameter, β is the scale parameter and $\Gamma(.)$ is the gamma function. The maximum likelihood function is given by

$$L(\underline{x})=\frac{1}{\beta^{n\,\alpha}}\prod_{i=1}^{n}\left(\frac{1}{\Gamma(\alpha)}X_i^{\alpha-1}\right)e^{-\frac{1}{\beta}\sum_{i=1}^{n}X_i}$$

and the derivative of its logarithm is

$$\frac{\partial \ln L(\underline{x})}{\partial \alpha}=-n\ln\beta-n\,\frac{\partial \ln\Gamma(\alpha)}{\partial\alpha}+\sum_{i=1}^{n}\ln X_i \tag{1}$$

by differentiating the above equation and equating to zero the estimate of α will be derived. To do so we need to find $\frac{\partial \ln\Gamma(\alpha)}{\partial\alpha}$.

Since for a fixed integer α , as the integer n increases, we have that (by Euler's definition as an infinite product)

$$\lim_{n\to\infty}\frac{n!(\mathrm{n}+1)^{\alpha}}{(\mathrm{n}+\alpha)!}=1.$$

It can be shown that

$$\Gamma(\alpha)=(\alpha-1)\,!$$

$$=\frac{1}{\alpha}\lim_{n\to\infty}\frac{n!\,\alpha!\,(\mathrm{n}+1)^{\alpha}}{(\mathrm{n}+\alpha)!}$$

$$=\frac{1}{\alpha}\lim_{n\to\infty}\frac{(1.2.3......n)\ \alpha!\ (\mathrm{n}+1)^{\alpha}}{\alpha!\ (\alpha+1)(\alpha+2).......(\mathrm{n}+\alpha)}$$

$$=\frac{1}{\alpha}\lim_{n\to\infty}\quad\prod_{k=1}^{n}\frac{\left(\frac{\mathrm{k}+1}{\mathrm{k}}\right)^{\alpha}}{\left(1+\frac{\alpha}{\mathrm{k}}\right)}$$

Therefore

$$\Gamma(\alpha)=\frac{1}{\alpha}\ \prod_{k=1}^{\infty}\frac{1}{\left(1+\frac{\alpha}{\mathrm{k}}\right)}\ \left(1+\frac{1}{\mathrm{k}}\right)^{\alpha}$$

And

$$\ln\Gamma(\alpha)=-\ln\alpha-\sum_{k=1}^{\infty}\ln\left(1+\frac{\alpha}{\mathrm{k}}\right)+\alpha\sum_{k=1}^{\infty}\ln\ \left(1+\frac{1}{\mathrm{k}}\right)$$

Thus

$$\frac{\partial\ln\Gamma(\alpha)}{\partial\alpha}=-\frac{1}{\alpha}-\sum_{k=1}^{\infty}\frac{1}{k+\alpha}+\sum_{k=1}^{\infty}\ln\ \left(1+\frac{1}{\mathrm{k}}\right)$$

$$=-\sum_{k=0}^{\infty}\frac{1}{k+\alpha}+\sum_{k=1}^{\infty}\ln\ \left(1+\frac{1}{\mathrm{k}}\right)\qquad(2)$$

Now by substitute (2) in (1)

$$\frac{\partial\ln L(\underline{x})}{\partial\alpha}=-n\ln\beta-n\left\{-\sum_{k=0}^{\infty}\frac{1}{k+\alpha}+\sum_{k=1}^{\infty}\ln\ \left(1+\frac{1}{\mathrm{k}}\right)\right\}+\sum_{i=1}^{n}\ln X_i$$

and let $\frac{\partial\ln L(\underline{x})}{\partial\alpha}=0$, so

$$\sum_{k=0}^{\infty}\frac{1}{k+\hat{\alpha}}=\ln\hat{\beta}+\sum_{k=1}^{\infty}\ln\ \left(1+\frac{1}{\mathrm{k}}\right)-\frac{1}{n}\sum_{i=1}^{n}\ln X_i$$

Since it is known that $\hat{\beta}=\frac{\overline{X}}{\hat{\alpha}}$ therefore

$$\sum_{k=0}^{n}\frac{1}{k+\hat{\alpha}}+\ln\hat{\alpha}=\ln\overline{X}+\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{\mathrm{k}}\right)-\frac{1}{n}\sum_{i=1}^{n}\ln X_i \qquad (3)$$

Now to find the estimate of α, regression equation estimates will be used.

Case 1:

Let $y=\sum_{k=0}^{n}\frac{1}{k+\hat{\alpha}}+\ln\hat{\alpha}$ be the response variable in (3), thus the estimated equation will be

$$\hat{y}=0.0803\ \frac{1}{\hat{\alpha}^2}+0.4939\ \frac{1}{\hat{\alpha}}+8.0099$$

with p-value =0.000 and $R^2=100\%$

Therefore $$0.0803\frac{1}{\hat{\alpha}^2}+0.4939\frac{1}{\hat{\alpha}}+8.0099-\ln\overline{X}-\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{\mathrm{k}}\right)+\frac{1}{n}\sum_{i=1}^{n}\ln X_i=0$$

Which can be written as

$$0.0803\ \hat{\alpha}^{*2}+0.4939\ \hat{\alpha}^{*}+c=0$$

Hence

$$\hat{\alpha}^{*}=\frac{-0.4939+\sqrt{(0.4939)^2-4*0.0803*c}}{2(0.0803)}$$

Where

$$c=8.0099-\ln\overline{X}-\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{\mathrm{k}}\right)+\frac{1}{n}\sum_{i=1}^{n}\ln X_i$$

Case 2

In equation (3) Let $y=\sum_{k=0}^{n}\frac{1}{k+\hat{\alpha}}+\ln\hat{\alpha}-\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{\mathrm{k}}\right)$ then the estimated regression equation will be given by $\hat{y}=0.5577\ \hat{\alpha}^{-1.027}$ with p-value=0.000 and $R^2=99.92\%$. Therefore

$$\hat{\alpha}=\left(\frac{\ln\overline{X}-\frac{1}{n}\sum_{i=1}^{n}\ln X_i}{0.5577}\right)^{-\frac{1}{1.027}}$$

Simulation study

This simulation study has been conducted to investigate the performance of the proposed method in estimating the gamma parameters. Samples of size n= 10, 50, 100 and 200 were simulated from the gamma distribution 1000 times with different parameters values (α, β)={(0.5,1),(1,1),(2,1)}.

Table (1) shows the estimates of the parameters along with its mean square errors for MM, MLE, MLEc, MLEcf methods and the proposed method. Also the sample root mean square error, RMSE, (Khamkong, 2018) was used for the comparison between all methods. As one can notice that the proposed method (case1) performed efficiently with smaller RMSE compared with other methods for different sample sizes. While case 2 performed better than MM and MLEcf. Generally, the RMSE decreased as the sample size increases for all methods. In addition as β increases the RMSE of case 1 method get closer to MLE.

Table 1: The estimates and the MSE's of the two parameter Gamma distribution for different sample sizes and different values of the parameters, along with RMSE for the method comparisons.

		$\alpha=0.5$	$\beta=1$	RMSE	$\alpha=1$	$\beta=1$	RMSE	$\alpha=2$	$\beta=1$	RMSE
n=10	MM	0.7575 (0.2484)	0.8771 (0.4508)	0.8362	1.4430 (0.8425)	0.8956 (0.3556)	1.0946	2.6697 (2.5258)	0.9456 (0.2902)	1.6781
	MLEcf	0.5877 (0.1355)	1.1218 (0.6213)	0.8700	1.3354 (0.7145)	0.9404 (0.3037)	1.0091	2.6934 (2.5598)	0.9142 (0.2171)	1.6664
	MLEc	0.6616 (0.1461)	0.9197 (0.3234)	0.6852	1.3864 (0.7340)	0.8827 (0.2500)	0.9920	2.7310 (2.6275)	0.8964 (0.2069)	1.6836
	MLE	0.6529 (0.1409)	0.9303 (0.3235)	0.6815	1.3749 (0.7234)	0.8887 (0.2476)	0.9854	2.7140 (2.5653)	0.9001 (0.2035)	1.6640
	Case1	0.6635 (0.1401)	0.9006 (0.2931)	0.6582	1.3780 (0.7387)	0.8852 (0.2414)	0.9900	2.7381 (2.7826)	0.8984 (0.2059)	1.7287
	Case2	0.6090 (0.1390)	1.0400 (0.4426)	0.7627	1.3632 (0.7765)	0.9229 (0.2901)	1.0328	2.7705 (2.8286)	0.8932 (0.2166)	1.7450
n=30	MM	0.6013 (0.0573)	0.9458 (0.2175)	0.5243	1.1518 (0.1733)	0.9645 (0.1425)	0.5620	2.1782 (0.4655)	1.0050 (0.1112)	0.7594
	MLEcf	0.4725 (0.0213)	1.1371 (0.2096)	0.4806	1.0520 (0.0788)	1.0098 (0.0968)	0.4190	2.1685 (0.3799)	0.9868 (0.0778)	0.6766
	MLEc	0.5460 (0.0212)	0.9608 (0.1120)	0.3650	1.1009 (0.0877)	0.9598 (0.0833)	0.4134	2.1886 (0.3836)	0.9761 (0.0742)	0.6766
	MLE	0.5420 (0.0200)	0.9648 (0.1097)	0.3601	1.0951 (0.0808)	0.9615 (0.0803)	0.4014	2.1914 (0.3815)	0.9737 (0.0732)	0.6743
	Case1	0.5542 (0.0200)	0.9381 (0.1007)	0.3474	1.0953 (0.0796)	0.9601 (0.0788)	0.3980	2.1950 (0.3950)	0.9735 (0.0742)	0.6849
	Case2	0.4924 (0.0191)	1.0760 (0.1559)	0.4184	1.0679 (0.0843)	0.9952 (0.0938)	0.4220	2.2249 (0.4325)	0.9639 (0.0774)	0.7141

		α=0.5	β=1	RMSE	α=1	β=1	RMSE	α=2	β=1	RMSE
n=50	MM	0.5696 (0.0335	0.9524 0.1414	0.4182	1.0909 (0.0832)	0.9776 (0.0878)	0.4135	2.1466 (0.2567)	0.9737 (0.0602)	0.5629
	MLEcf	0.4550 (0.0115	1.1381 0.1342	0.3818	1.0067 (0.0419)	1.0330 (0.0644)	0.3260	2.0971 (0.1812)	0.9833 (0.0451)	0.4757
	MLEc	0.5281 (0.0099	0.9670 0.0687	0.2804	1.0552 (0.0437)	0.9812 (0.0528)	0.3106	2.1250 (0.1875)	0.9692 (0.0431)	0.4802
	MLE	0.5253 (0.0089	0.9703 0.0679	0.2772	1.0508 (0.0419)	0.9842 (0.0519)	0.3061	2.1201 (0.1825)	0.9709 (0.0427)	0.4746
	Case1	0.5378 (0.0091	0.9444 0.0633	0.2691	1.0510 (0.0411)	0.9832 (0.0508)	0.3032	2.1216 (0.1873)	0.9709 (0.0433)	0.4803
	Case2	0.4749 (0.0092)	1.0813 0.0987	0.3285	1.0209 (0.0441)	1.0189 (0.0619)	0.3256	2.1502 (0.2094)	0.9602 (0.0455)	0.5050
n=100	MM	0.5347 (0.0150)	0.9895 (0.0773)	0.3038	1.0455 (0.0377)	0.9907 (0.0424)	0.2830	2.0824 (0.1267)	0.9858 (0.0328)	0.3994
	MLEcf	0.4425 (0.0080)	1.1624 (0.0837)	0.3028	0.9752 (0.0171)	1.0450 (0.0295)	0.2157	2.0350 (0.0795)	0.9996 (0.0235)	0.3209
	MLEc	0.5145 (0.0046)	0.9923 (0.0344)	0.1975	1.0218 (0.0167)	0.9957 (0.0232)	0.1999	2.0625 (0.0823)	0.9858 (0.0226)	0.3238
	MLE	0.5134 (0.0043)	0.9932 (0.0330)	0.1931	1.0198 (0.0158)	0.9969 (0.0225)	0.1957	2.0584 (0.0799)	0.9874 (0.0222)	0.3195
	Case1	0.5263 (0.0044)	0.9671 (0.0305)	0.1870	1.0201 (0.0154)	0.9962 (0.0221)	0.1937	2.0586 (0.0815)	0.9876 (0.0225)	0.3226
	Case2	0.4625 (0.0056)	1.1068 (0.0573)	0.2508	0.9881 (0.0172)	1.0314 (0.0278)	0.2121	2.0854 (0.0924)	0.9761 (0.0237)	0.3407
n=200	MM	0.5187 (0.0076)	0.9877 (0.0380)	0.2136	1.0268 (0.0202)	0.9945 (0.0245)	0.2113	2.0315 (0.0599)	0.9971 (0.0184)	0.2798
	MLEcf	0.4351 (0.0064)	1.1583 (0.0535)	0.2448	0.9684 (0.0098)	1.0446 (0.0176)	0.1658	2.0039 (0.0379)	1.0053 (0.0123)	0.2241
	MLEc	0.5069 (0.0020)	0.9906 (0.0175)	0.1397	1.0146 (0.0090)	0.9960 (0.0133)	0.1494	2.0278 (0.0386)	0.9933 (0.0120)	0.2248
	MLE	0.5065 (0.0019)	0.9911 (0.0171)	0.1380	1.0131 (0.0084)	0.9971 (0.0129)	0.1459	2.0275 (0.0377)	0.9931 (0.0116)	0.2221
	Case1	0.5195 (0.0021)	0.9654 (0.0166)	0.1367	1.0133 (0.0082)	0.9966 (0.0126)	0.1443	2.0271 (0.0384)	0.9935 (0.0118)	0.2240
	Case2	0.4553 (0.0039)	1.1045 (0.0342)	0.1953	0.9809 (0.0095)	1.0312 (0.0163)	0.1605	2.0529 (0.0440)	0.9817 (0.0124)	0.2375

Conclusion

The Gamma distribution is an important distribution which is widely used in many fields. Hence, estimating its parameters is an important issue. There are several methods to estimate these parameters in the literature. In this work we proposed an alternative method (case 1 and case 2) to estimate the two parameter gamma distribution. One can observe that the proposed method (case 1) performed better than the other methods presented in Table 1. While case 2 performed better than MM and MLEcf methods and being closed to the rest.

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